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NUMERICAL ANALYSIS OF CONTINUOUS FRAMES IN SPACE

by James Michalos, M. ASCE

ENGINEERING MECHANICS DIVISION

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PAPERS

NUMERICAL ANALYSIS OF CONTINUOUS FRAMES IN SPACE

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SYNOPSIS

A numerical procedure, involving successive corrections, is presented for determining moments in continuous girders and frames whose members join at arbitrary angles in space. The component parts of these structures may be of constant or variable cross section.

INTRODUCTION

An analytical investigation of the structural behavior of a continuous building frame is usually made by neglecting the torsional resistance of girders at right angles to the particular bent under study. Usually, this simplification has not caused great concern among designers. The effect of these girders, and the effect on these girders, can be appraised, providing one knows—or can approximate—the relationship between angle of twist and torque for such members. An analysis that includes the torsional resistance of members at right angles to a frame can be made by application of the ordinary method of moment distribution[2] to all members framing into each joint. This has been explained elsewhere.[3,4,5]

[2] "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions*, ASCE, Vol. 97, 1932, p. 1.

[3] "Design of Reinforced Concrete in Torsion," by Paul Andersen, *Ibid.*, Vol. 103, 1938, p. 1503.

[4] "Reinforced Concrete Structures," by Dean Peabody, Jr., John Wiley & Sons, Inc., New York, N. Y., 2d Ed., 1946, p. 419.

[5] "Analysis of Three-Dimensional Beam and Girder Framing," by Phil M. Ferguson, *Journal*, A.C.I., Vol. 22, September, 1950, p. 61.

Structures may be built whose members frame into each other at various angles in space rather than at right angles. It seems probable that requirements of clearance, space, and architecture will accelerate the use of structures having nonorthogonal framing, not only in building construction but also in connection with urban expressway construction. Simplifications are neither as obvious nor as acceptable as in those cases in which only relatively small

moments are introduced into the joint by members at right angles to a bent.

In this paper a numerical procedure is presented for analyzing girders and frames that are continuous in space. It is limited herein to straight members, but these may be of constant or variable section. Furthermore, the members may be oriented in space in any arbitrary fashion, both with respect to their longitudinal axes and the principal axes of their cross sections. In general, the paper deals with structures in which there is no joint translation, or in which such movement is negligible, but the handling of cases in which translation occurs is outlined and illustrated by means of a numerical example.

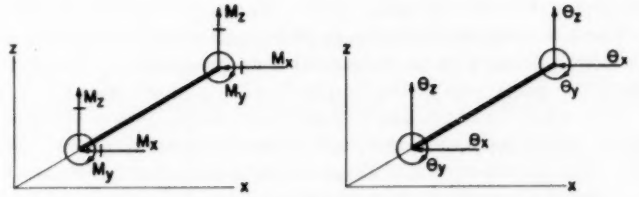
The method involves successive corrections and can be carried to any desired precision. It is an extension into space of the philosophy of moment distribution, and it is based on a concept wherein all defined properties and moments are referred to a system of rectangular axes.

Expressions for the physical constants used in the numerical procedure are derived for the general case of a member in space subjected to moments about each of three rectangular axes, none of which is parallel to the longitudinal axis of the member or to the principal axes of the cross section. Simplified expressions are obtained for several special, or more usual, cases. The numerical procedure is then outlined, and examples are presented.

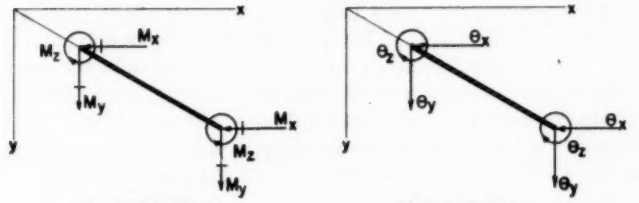
Stiffness and Carry-Over Factors

When the method of moment distribution is used in the analysis of a frame lying in a plane, the unbalanced moment at each joint is distributed to all members framing into that joint. The portion of the unbalanced moment taken by each member is in proportion to its "stiffness," which is defined as the moment necessary to rotate one end of a member through a unit angle when the other end is fixed and there is no relative displacement of the ends of the member. But a distributed moment, resulting from rotation at one end of a member, induces moment at the other (fixed) end. This moment is determined by multiplying the distributed moment by a "carry-over factor," which is defined as the ratio of the moment induced at a fixed end to the moment producing rotation at the other end.

In the method presented in this paper, all moments are referred to a system of three rectangular axes, and a joint is considered free to rotate about one axis at a time. Consider a member that is not parallel to any of the three orthogonal planes associated with an arbitrarily chosen system of rectangular axes. If moment is applied about one of the reference axes at one end—and if rotation is prevented about the other two axes at that end and all three reference axes at the other end—carry-over moments will be induced at each end about the axes of no rotation. Stiffness factors will be required for rotation about each of the three reference axes. Five carry-over factors will be required for each of the three axes; two for the end at which rotation has occurred, and three for the opposite end. The adopted sign convention is shown in Fig. 1, wherein positive end moments and rotations are represented vectorially. The reader should use the right-hand rule for the sense of moments and rotations. Thus, if the thumb of the right hand is pointed in the direction of the arrow, the curved fingers indicate the direction of the moment or rotation.



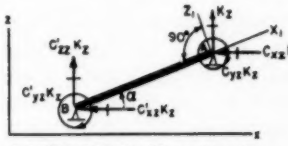
Note: Use right-hand rule for sense of moments and rotations



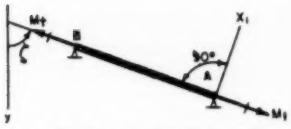
(a) MOMENTS (b) ROTATIONS

FIG. 1.—POSITIVE MOMENTS AND ROTATIONS AT THE ENDS OF A MEMBER

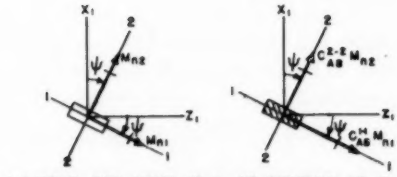
Several views of a member supported at both ends are shown in Fig. 2. In addition to an arbitrarily chosen set of rectangular axes, x, y, and z, an auxiliary pair of axes, x_1 and z_1, is introduced, but only as an aid in the derivation of



(a) PROJECTION ON xz PLANE

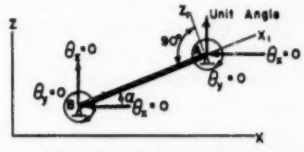


(b) TRUE LONGITUDINAL VIEW

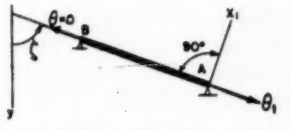


(c) TRUE END VIEW AT A (d) SECTION AT B

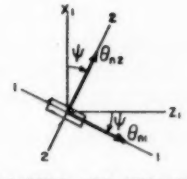
FIG. 2.—MOMENT RELATIONSHIPS



(a) PROJECTION ON xz PLANE



(b) TRUE LONGITUDINAL VIEW



(c) TRUE END VIEW AT A

FIG. 3.—ROTATION RELATIONSHIPS

expressions for stiffness and carry-over. They have no role in the procedure itself. The z_1-axis is parallel to the xz-plane, and the x_1-axis forms a right angle with the longitudinal axis of the member, its projection in the xz-plane coinciding with the projection of the longitudinal axis of the member in the same plane. The axis 1–1 and the axis 2–2 are the principal axes of the cross section. The angles ψ and ζ are true angles, and α is the projection of an angle upon the xz-plane. The positive sense of these angles is indicated by arrow.

If a moment K_z applied about the z-axis at end A of the member in Fig. 2(a) produces a unit rotation about the same axis, while no rotation is allowed to take place about the x-axis and the y-axis at end A or about any of the axes at end B, the moments necessary to prevent rotation about these axes can be written as shown in Fig. 2(a). The moment K_z is the defined stiffness about the z-axis at end A, and C_{xz} and C_{yz} are carry-over factors to the x-axis and the y-axis, respectively, at the same end, whereas C'_{xz}, C'_{xz}, and C'_{yz} are carry-over factors to the far end, B. The contributions of the moments K_z, C_{xz}, K_z, and $C_{yz}\,K_z$ to torsional moment and to bending moments about principal axes at end A, as shown in Figs. 2(b) and 2(c), can be obtained as follows:

$$\left.\begin{aligned} M_t &= K_z \sin\alpha \sin\zeta - C_{xz}\,K_z \cos\alpha \sin\zeta + C_{yz}\,K_z \cos\zeta \\ M_{n1} &= K_z \cos\alpha \cos\psi - K_z \sin\alpha \cos\zeta \sin\psi + C_{xz}\,K_z \sin\alpha \cos\psi \\ &\quad + C_{xz}\,K_z \cos\alpha \cos\zeta \sin\psi + C_{yz}\,K_z \sin\zeta \sin\psi \\ M_{n2} &= K_z \cos\alpha \sin\psi + K_z \sin\alpha \cos\zeta \cos\psi + C_{xz}\,K_z \sin\alpha \sin\psi \\ &\quad - C_{xz}\,K_z \cos\alpha \cos\zeta \cos\psi - C_{yz}\,K_z \sin\zeta \cos\psi \end{aligned}\right\} \ldots (1)$$

The bending moments induced about the principal axes at B are $C^{1-1}{}_{AB}\,M_{n1}$ and $C^{2-2}{}_{AB}\,M_{n2}$, as shown in Fig. 2(d). The terms $C^{1-1}{}_{AB}$ and $C^{2-2}{}_{AB}$ are carry-over factors, from A to B, with respect to axis 1–1 and axis 2–2, respectively. The value of these carry-over factors is 0.5 for prismatic members. Values for nonprismatic members can be obtained from available charts or tables.[6]

[6] "Handbook of Frame Constants," Portland Cement Assn., Chicago, Ill., 1947.

In Fig. 3(a) the end A of the member has been rotated through a unit angle about the z-axis only. The contributions of the unit angle to twist and to rotation about principal axes at end A, as shown in Figs. 3(b) and 3(c), can be obtained as follows:

$$\left.\begin{aligned} \theta_t &= \sin\alpha \sin\zeta \\ \theta_{n1} &= \cos\alpha \cos\psi - \sin\alpha \cos\zeta \sin\psi \\ \theta_{n2} &= \cos\alpha \sin\psi + \sin\alpha \cos\zeta \cos\psi \end{aligned}\right\} \ldots\ldots\ldots\ldots (2)$$

This angle of twist and these rotation angles can be related to the moments of Eq. 1 by means of stiffnesses k_{n1}, k_{n2}, and k_t. The terms k_{n1} and k_{n2} represent the bending stiffnesses associated with moment distribution. Their values, for prismatic members, are $\frac{4\,E\,I_1}{L}$ and $\frac{4\,E\,I_2}{L}$, respectively. In these expressions, E is the modulus of elasticity in tension and compression, L is the length of the member, and I_1 and I_2 are the moments of inertia about axis 1–1 and axis 2–2,

respectively. For nonprismatic members, values of k_{n1} and k_{n2} are available elsewhere.[6] The term k_t represents the moment necessary to twist one end of a member through a unit angle when the other end is fixed. Its value for prismatic members is $\frac{G\,J}{L}$, G being the modulus of elasticity in shear and J being a torsion factor, such that $G\,J$ represents the torque necessary to twist a member through a unit angle per unit length. The value of k_t for nonprismatic members can be determined through the use of the shear and torsion analogy.[7]

[7] "Laterally Loaded Plane Structures and Structures Curved in Space," by Frank Baron and James P. Michalos, *Transactions*, ASCE, Vol. 117, 1952, p. 279.

Three equations, relating end moments and end rotations about the longitudinal axis and the principal axes, can be written as follows:

$$\frac{1}{k_t}\,(\sin\alpha\sin\zeta\,K_z - \cos\alpha\sin\zeta\,C_{xz}\,K_x + \cos\zeta\,C_{yz}\,K_z) = \sin\alpha\sin\zeta\,. .\,(3a)$$

$$\frac{1}{k_{n1}}\,[(\cos\alpha\cos\psi - \sin\alpha\cos\zeta\sin\psi)\,K_z + (\sin\alpha\cos\psi + \cos\alpha\cos\zeta \times \sin\psi)\,C_{xz}\,K_x + (\sin\zeta\sin\psi)\,C_{yz}\,K_z] = \cos\alpha\cos\psi - \sin\alpha\cos\zeta\sin\psi\,. .\,(3b)$$

$$\frac{1}{k_{n2}}\,[(\cos\alpha\sin\psi + \sin\alpha\cos\zeta\cos\psi)\,K_z + (\sin\alpha\sin\psi - \cos\alpha\cos\zeta\cos\psi)\,C_{xz}\,K_z - (\sin\zeta\cos\psi)\,C_{yz}\,K_z] = \cos\alpha\sin\psi + \sin\alpha\cos\zeta\cos\psi\,. .\,(3c)$$

In an identical manner, similar groups of equations are obtained as the result of applying a moment K_x about the x-axis and a moment K_y about the y-axis in Fig. 2(*a*).

From the application of K_x—

$$\frac{1}{k_t}\,(\cos\alpha\sin\zeta\,K_x - \sin\alpha\sin\zeta\,C_{zx}\,K_x - \cos\zeta\,C_{yx}\,K_x) = \cos\alpha\sin\zeta\,. .\,(4a)$$

$$\frac{1}{k_{n1}}\,[(\sin\alpha\cos\psi + \cos\alpha\sin\psi\cos\zeta)\,K_x + (\cos\alpha\cos\psi - \sin\alpha\sin\psi\cos\zeta)\,C_{zx}\,K_x + \sin\psi\sin\zeta\,C_{yx}\,K_x] = \sin\alpha\cos\psi + \cos\alpha\sin\psi\cos\zeta\,. .\,(4b)$$

and

$$\frac{1}{k_{n2}}\,[(\sin\alpha\sin\psi - \cos\alpha\cos\psi\cos\zeta)\,K_x + (\cos\alpha\sin\psi + \sin\alpha\cos\psi\cos\zeta)\,C_{zx}\,K_x - \cos\psi\sin\zeta\,C_{yx}\,K_x] = \sin\alpha\sin\psi - \cos\alpha\cos\psi\cos\zeta\,. .\,(4c)$$

From the application of K_y—

$$\frac{1}{k_t}(\cos\zeta\,K_y + \sin\alpha\sin\zeta\,C_{xy}\,K_y - \cos\alpha\sin\zeta\,C_{zy}\,K_y) = \cos\zeta\,. . . .\,(5a)$$

$$\frac{1}{k_{n1}}\,[\sin\psi\sin\zeta\,K_y + (\cos\alpha\cos\psi - \sin\alpha\sin\psi\cos\zeta)\,C_{xy}\,K_y + (\sin\alpha\cos\psi + \cos\alpha\sin\psi\cos\zeta)\,C_{zy}\,K_y] = \sin\psi\sin\zeta\,. .\,(5b)$$

and

$$\frac{1}{k_{n2}}[-\cos\psi\sin\zeta\,K_y + (\cos\alpha\sin\psi + \sin\alpha\cos\psi\cos\zeta)\,C_{zy}\,K_y + (\sin\alpha\sin\psi - \cos\alpha\cos\psi\cos\zeta)\,C_{xy}\,K_y] = -\cos\psi\sin\zeta \quad . . (5c)$$

From the solution of the three groups of simultaneous equations (Eqs. 3, 4, and 5) expressions for the three stiffnesses (K_z, K_y, and K_x) and the six near-end carry-over factors (C_{xz}, C_{yz}, C_{yx}, C_{zx}, C_{zy}, and C_{xy}) were obtained and are shown at the top of Fig. 4. The coefficients, N, have been introduced for convenience, and they are defined in Fig. 5. These coefficients are functions of the angles only, and some of the expressions are greatly simplified—as shown in the bottom half of Fig. 5—in the usual case in which the angle ψ is equal to zero.

The carry-over moments at end B of Fig. 2(*a*) can be written in terms of the torsional and bending moments at the same end as follows:

$$\left.\begin{aligned} C'_{xz}M_z &= -M_t\sin\alpha\sin\zeta + C^{1-1}{}_{AB}M_{n1}(\cos\psi\cos\alpha - \sin\psi\cos\zeta \times \sin\alpha) + C^{2-2}{}_{AB}M_{n2}(\sin\psi\cos\alpha + \cos\psi\cos\zeta\sin\alpha) \\ C'_{yz}M_z &= -M_t\cos\zeta + C^{1-1}{}_{AB}M_{n1}\sin\psi\sin\zeta - C^{2-2}{}_{AB}M_{n2}\cos\psi\sin\zeta \\ C'_{zz}M_z &= M_t\sin\zeta\cos\alpha + C^{1-1}{}_{AB}M_{n1}(\cos\psi\sin\alpha + \sin\psi\cos\zeta \times \cos\alpha + C^{2-2}{}_{AB}M_{n2}(\sin\psi\sin\alpha - \cos\psi\cos\zeta\cos\alpha) \end{aligned}\right\} \ldots . (6)$$

If the right-hand sides of Eqs. 1 are substituted for M_t, M_{n1}, and M_{n2} in Eqs. 6, expressions for the far-end carry-over factors can be obtained in terms of the two carry-over factors at A and the two carry-over factors that are related to the principal axes. The resulting expressions for C'_{xz}, C'_{zz}, and C'_{yz} are given in Fig. 4, in terms of coefficients, N, which are defined in Fig. 5.

$$K_z = k_t N_4 - k_{n1}(N_5 - N_7) + k_{n2}(N_5 + N_6) \qquad C_{xz} = \frac{-k_t N_1 + k_{n1}(N_2 + N_{22}) - k_{n2}(N_2 - N_3)}{K_z} \qquad C_{zx} = C_{xz}\frac{K_z}{K_x}$$

$$K_x = k_t N_{12} + k_{n1}(N_8 + N_{14}) - k_{n2}(N_5 - N_{13}) \qquad C_{yz} = \frac{k_t N_8 + k_{n1}(N_9 - N_{11}) - k_{n2}(N_9 + N_{10})}{K_z} \qquad C_{zy} = C_{yz}\frac{K_z}{K_y}$$

$$K_y = k_t N_{19} + k_{n1}N_{21} + k_{n2}N_{20} \qquad C_{yx} = \frac{-k_t N_{18} + k_{n1}(N_{17} + N_{16}) + k_{n2}(N_{16} - N_{17})}{K_x} \qquad C_{xy} = C_{yx}\frac{K_x}{K_y}$$

$$C'_{xz} = N_1 - C_{xz}N_{12} + C_{yz}N_{15} + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_2 + C_{xz}N_5 + C_{yz}N_{17}) + C^{1-1}_{A-B}(N_{22} + C_{xz}N_{14} + C_{yz}N_{16}) + C^{2-2}_{A-B}(N_3 + C_{xz}N_{13} + C_{yz}N_{16})$$

$$C'_{zz} = -N_4 + C_{xz}N_1 - C_{yz}N_8 + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(-N_5 + C_{xz}N_2 + C_{yz}N_9) + C^{1-1}_{A-B}(N_7 + C_{xz}N_{22} - C_{yz}N_{11}) + C^{2-2}_{A-B}(N_6 + C_{xz}N_3 + C_{yz}N_{10})$$

$$C'_{yz} = -N_8 + C_{xz}N_{18} - C_{yz}N_{19} + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_9 + C_{xz}N_{17}) + C^{1-1}_{A-B}(-N_{11} + C_{xz}N_{18} + C_{yz}N_{21}) + C^{2-2}_{A-B}(-N_{10} + C_{xz}N_{16} + C_{yz}N_{20})$$

$$C'_{xx} = -N_{12} + C_{zx}N_1 + C_{yx}N_{15} + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_5 + C_{zx}N_2 + C_{yx}N_{17}) + C^{1-1}_{A-B}(N_{14} + C_{zx}N_{22} - C_{yx}N_{16}) + C^{2-2}_{A-B}(N_{13} + C_{zx}N_3 + C_{yx}N_{16})$$

$$C'_{zx} = N_1 - C_{zx}N_4 - C_{yx}N_8 + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_2 - C_{zx}N_5 + C_{yx}N_9) + C^{1-1}_{A-B}(N_{22} + C_{zx}N_7 - C_{yx}N_{11}) + C^{2-2}_{A-B}(N_3 + C_{zx}N_6 - C_{yx}N_{10})$$

$$C'_{yx} = N_{19} - C_{zx}N_8 - C_{yx}N_{19} + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_{17} + C_{zx}N_9) + C^{1-1}_{A-B}(N_{16} - C_{zx}N_{11} + C_{yx}N_{21}) + C^{2-2}_{A-B}(N_{16} - C_{zx}N_{10} + C_{yx}N_{20})$$

$$C'_{xy} = N_{16} - C_{zy}N_{12} + C_{zy}N_1 + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_{17} + C_{xy}N_5 + C_{zy}N_2) + C^{1-1}_{A-B}(N_{18} + C_{xy}N_{14} + C_{zy}N_{22}) + C^{2-2}_{A-B}(N_{18} + C_{xy}N_{13} + C_{zy}N_3)$$

$$C'_{zy} = -N_8 + C_{zy}N_1 - C_{zy}N_4 + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(N_9 + C_{xy}N_2 - C_{zy}N_5) + C^{1-1}_{A-B}(-N_{11} + C_{xy}N_{22} + C_{zy}N_7) + C^{2-2}_{A-B}(-N_{10} + C_{xy}N_3 + C_{zy}N_6)$$

$$C'_{yy} = -N_{19} + C_{xy}N_{18} - C_{zy}N_8 + (C^{1-1}_{A-B} - C^{2-2}_{A-B})(C_{xy}N_{17} + C_{zy}N_9) + C^{1-1}_{A-B}(N_{21} + C_{xy}N_{16} - C_{zy}N_{11}) + C^{2-2}_{A-B}(N_{20} + C_{xy}N_{16} - C_{zy}N_{10})$$

Fig. 4.—General Expressions for Stiffness and Carry-Over Factors

For the general case of a member in space:

$N_1 = \sin\alpha \cos\alpha \sin^2\zeta$

$N_2 = (\cos^2\alpha - \sin^2\alpha)\sin\psi \cos\psi \cos\zeta$

$N_3 = \sin\alpha \cos\alpha (\sin^2\psi - \cos^2\psi \cos^2\zeta)$

$N_4 = \sin^2\alpha \sin^2\zeta$

$N_5 = 2 \sin\alpha \cos\alpha \sin\psi \cos\psi \cos\zeta$

$N_6 = \sin^2\alpha \cos^2\psi \cos^2\zeta + \cos^2\alpha \sin^2\psi$

$N_7 = \sin^2\alpha \sin^2\psi \cos^2\zeta + \cos^2\alpha \cos^2\psi$

$N_8 = \sin\alpha \sin\zeta \cos\zeta$

$N_9 = \cos\alpha \sin\psi \cos\psi \sin\zeta$

$N_{10} = \sin\alpha \cos^2\psi \sin\zeta \cos\zeta$

$N_{11} = \sin\alpha \sin^2\psi \sin\zeta \cos\zeta$

$N_{12} = \cos^2\alpha \sin^2\zeta$

$N_{13} = \cos^2\alpha \cos^2\psi \cos^2\zeta + \sin^2\alpha \sin^2\psi$

$N_{14} = \cos^2\alpha \sin^2\psi \cos^2\zeta + \sin^2\alpha \cos^2\psi$

$N_{15} = \cos\alpha \sin\zeta \cos\zeta$

$N_{16} = \cos\alpha \cos^2\psi \sin\zeta \cos\zeta$

$N_{17} = \sin\alpha \sin\psi \cos\psi \sin\zeta$

$N_{18} = \cos\alpha \sin^2\psi \sin\zeta \cos\zeta$

$N_{19} = \cos^2\zeta$

$N_{20} = \cos^2\psi \sin^2\zeta$

$N_{21} = \sin^2\psi \sin^2\zeta$

$N_{22} = \sin\alpha \cos\alpha (\cos^2\psi - \sin^2\psi \cos^2\zeta)$

When $\psi = 0$:

$N_2 = 0$

$N_3 = -\sin\alpha \cos\alpha \cos^2\zeta$

$N_5 = 0$

$N_6 = \sin^2\alpha \cos^2\zeta$

$N_7 = \cos^2\alpha$

$N_9 = 0$

$N_{10} = \sin\alpha \sin\zeta \cos\zeta$

$N_{11} = 0$

$N_{13} = \cos^2\alpha \cos^2\zeta$

$N_{14} = \sin^2\alpha$

$N_{16} = \cos\alpha \sin\zeta \cos\zeta$

$N_{17} = 0$

$N_{18} = 0$

$N_{20} = \sin^2\zeta$

$N_{21} = 0$

$N_{22} = \sin\alpha \cos\alpha$

Fig. 5.—Coefficients for Use in the Expressions of Fig. 4

When: $\zeta = 90°$ and $\psi = 0$:-

$K_z = k_t \sin^2\alpha + k_{n1} \cos^2\alpha$

$K_x = k_t \cos^2\alpha + k_{n1} \sin^2\alpha$

$K_y = k_{n2}$

$C_{xz} = \dfrac{(k_{n1} - k_t)\sin\alpha \cos\alpha}{K_z}$

$C_{zx} = \dfrac{(k_{n1} - k_t)\sin\alpha \cos\alpha}{K_x}$

$C_{yz} = 0$; $C_{yx} = 0$

$C_{zy} = 0$; $C_{xy} = 0$

$C'_{xz} = (C_{AB}^{1-1} + 1)\sin\alpha \cos\alpha + C_{xz}(C_{AB}^{1-1} \sin^2\alpha - \cos^2\alpha)$

$C'_{zz} = C_{AB}^{1-1} \cos^2\alpha - \sin^2\alpha + C_{xz} \sin\alpha \cos\alpha (C_{AB}^{1-1} + 1)$

$C'_{xx} = C_{AB}^{1-1} \sin^2\alpha - \cos^2\alpha + C_{zx} \sin\alpha \cos\alpha (C_{AB}^{1-1} + 1)$

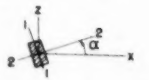
$C'_{zx} = (C_{AB}^{1-1} + 1)\sin\alpha \cos\alpha + C_{zx}(C_{AB}^{1-1} \cos^2\alpha - \sin^2\alpha)$

$C'_{yz} = 0$; $C'_{yx} = 0$

$C'_{zy} = 0$; $C'_{xy} = 0$

$C'_{yy} = C_{AB}^{2-2}$

When: $\zeta = 0$ ($\therefore \psi = 0$) :-



$K_z = k_{n2} \sin^2\alpha + k_{n1} \cos^2\alpha$

$K_x = k_{n2} \cos^2\alpha + k_{n1} \sin^2\alpha$

$K_y = k_t$

$C_{xz} = \dfrac{(k_{n1} - k_{n2})\sin\alpha \cos\alpha}{K_z}$

$C_{zx} = \dfrac{(k_{n1} - k_{n2})\sin\alpha \cos\alpha}{K_x}$

$C_{yz} = 0$; $C_{yx} = 0$

$C_{zy} = 0$; $C_{xy} = 0$

$C'_{xz} = (C_{AB}^{1-1} - C_{AB}^{2-2})\sin\alpha \cos\alpha + C_{xz}(C_{AB}^{1-1} \sin^2\alpha + C_{AB}^{2-2} \cos^2\alpha)$

$C'_{zz} = (C_{AB}^{1-1} - C_{AB}^{2-2}) C_{xz} \sin\alpha \cos\alpha + C_{AB}^{1-1} \cos^2\alpha + C_{AB}^{2-2} \sin^2\alpha$

$C'_{xx} = (C_{AB}^{1-1} - C_{AB}^{2-2}) C_{zx} \sin\alpha \cos\alpha + C_{AB}^{1-1} \sin^2\alpha + C_{AB}^{2-2} \cos^2\alpha$

$C'_{zx} = (C_{AB}^{1-1} - C_{AB}^{2-2})\sin\alpha \cos\alpha + C_{zx}(C_{AB}^{1-1} \cos^2\alpha + C_{AB}^{2-2} \sin^2\alpha)$

$C'_{yz} = 0$; $C'_{yx} = 0$

$C'_{zy} = 0$; $C'_{xy} = 0$

$C'_{yy} = -1$

Note: For prismatic members, $C^{1-1} = C^{2-2} = 0.5$

Fig. 6 —Simplified Expressions for Use in Common Cases

Similarly, the application of a moment K_x, instead of K_z, at A in Fig. 2(*a*) would lead to expressions for the three far-end carry-over factors C'_{xx}, C'_{zx}, and C'_{yx}, and the application of a moment K_y at A would lead to expressions for C'_{xy}, C'_{zy}, and C'_{yy}. The expressions for these carry-over factors are included in Fig. 5. In each case, the first subscript indicates the axis to which moment is carried, and the second subscript indicates the axis about which moment is applied.

The expressions for stiffness and for carry-over factors, as shown in Fig. 4, are for the general case. However, for the majority of cases which may be encountered in practice, the expressions are much simplified, as shown in the bottom half of Fig. 5 and in Fig. 6.

The Numerical Procedure

Before the numerical procedure can be applied, it is necessary to compute the stiffnesses and the carry-over factors. In the great majority of cases, ψ is equal to zero, and the expressions are greatly simplified. If, in addition, ζ is equal to 90° or to zero, the expressions in Fig. 6 are directly applicable.

Only relative values of k_{n1}, k_{n2}, and k_t are necessary, and for prismatic members both $C'^{1-1}{}_{AB}$ and $C'^{2-2}{}_{AB}$ are equal to 0.5, thus further simplifying the expressions for carry-over factors. In dealing with prismatic members, or with symmetrically haunched members, the required stiffness values are the same for each end, but otherwise two sets of stiffness values must be determined for each member, one set for each end.

The application of the numerical procedure can be outlined in the following steps:

1. The fixed-end moments, about the principal axes, are computed for each end of all loaded members.
2. These moments are resolved to fixed-end moments about the rectangular axes of the selected reference system. In general, this will result in unbalanced moments about the reference axes at each joint.
3. At each joint, the unbalanced moment about each axis is distributed, in turn, to all members framing into the joint in proportion to their relative stiffnesses about each of the axes. The distribution at each joint is algebraic, and signs are determined automatically.
4. Each distributed moment is multiplied (with due regard to signs), by the appropriate carry-over factors in turn, and the resulting moments are carried over to the proper axes at both the near and far ends.
5. Steps 3 and 4 are repeated until a desired convergence is reached.
6. All values following Step 2 are totalled algebraically to obtain the end moments about the reference axes.
7. The values obtained in Step 6 are resolved into bending moments about the principal axes and into torsional moments.

The girder shown in Fig. 7 is fixed at ends A and C, and is continuous over support B. Members AB and BC, both of which lie in the *xz*-plane, are identical in length and cross section, with relative bending stiffness about axis 1–1 equal to ten, and relative torsional stiffness equal to one. Since the load is applied normal to the *xz*-plane, and the angle ψ is equal to zero, there will be no

moments about the y-axis. As a result of this, it is not necessary to determine stiffnesses or carry-over factors associated with that axis. The required constants are readily determined from the simplified expressions ($\psi = 0; \zeta = 90°$) in Fig. 6. Ends A and C are fixed, and there will be no carry-over from A to B or from C to B.

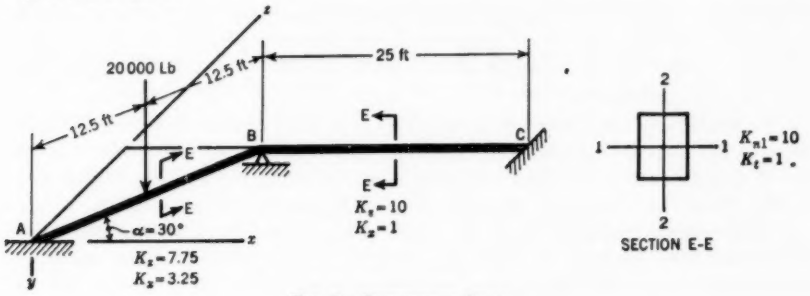


FIG. 7.—CONTINUOUS BEAM

TABLE 1.—ANALYSIS OF THE CONTINUOUS BEAM OF FIG. 7.

DESCRIPTION[a]	MOMENTS ABOUT THE x-AXIS, IN POUND-FEET				MOMENTS ABOUT THE z-AXIS, IN POUND-FEET			
	Joint A	Joint B		Joint C	Joint A	Joint B		Joint C
	Member AB	Member BA	Member BC	Member CB	Member AB	Member BA	Member BC	Member CB
Factor:								
Dist.		+0.765	+0.235			+0.437	+0.563	
C.O.		C'_{xx} = +0.154 C'_{zx} = +0.799 C_{xx} = +1.199	C'_{zz} = −1.0			C'_{xx} = +0.452 C'_{zx} = +0.335 C_{zz} = +0.503	C'_{zz} = +0.5	
F.E.M.	−31,250	+31,250			−54,125	+54,125		
Dist.		−23,906	−7,344			−23,653	−30,472	
COM (x)	−3,682			+7,344	−19,101	−28,663		
COM (z)	−7,924	−11,897			−10,691			−15,236
Dist.		+9,101	+2,796			+12,526	+16,137	
COM (x)	+1,402			−2,796	+7,272	+10,912		
COM (z)	+4,196	+6,301			+5,662			+8,068
Dist.		−4,820	−1,481			−4,769	−6,143	
COM (x)	−742			+1,481	−3,851	−5,779		
COM (z)	−1,598	−2,399			−2,156			−3,072
Dist.		+1,835	+564			+2,525	+3,254	
COM (x)	+283			−564	+1,466	+2,200		
COM (z)	+846	+1,270			+1,141			+1,627
Dist.		−972	−298			−961	−1,239	
COM (x)	−150			+298	−777	−1,165		
COM (z)	−322	−483			−434			−620
Dist.		+369	+114			+509	+656	
COM (x)	+57			−114	+295	+442		
COM (z)	+171	+256			+230			+328
Dist.		−196	−60			−193	−249	
COM (x)	−30			+60	−157	−235		
COM (z)	−64	−97			−87			−124
Dist.		+74	+23			+103	+132	
COM (x)	+11			−23	+59	+89		
COM (z)	+34	+52			+46			+66
Dist.		−40	−12			−39	−50	
COM (x)	−6			+12	−32	−48		
COM (z)	−13	−20			−18			−25
Dist.		+15	+5			+21	+27	
Total	−38,781	+5,693	−5,693	+5,698	−75,258	+17,947	−17,947	−8,988

[a] Abbreviations: Dist. = Distribution; C.O. = Carry-Over; F.E.M. = Fixed-End Moment; COM (x) = Carry-Over Moment from the x-Axis; COM (z) = Carry-Over Moment from the z-Axis.

It can be seen by inspection that for member BC, K_x is equal to k_{s1}, and K_z is equal to k_t. Likewise, the carry-over factors from B to C are C_{xx} (equal to -1.0) and C'_{zz} (equal to $+0.5$). For convenience in distributing unbalanced moments, the ratio of the stiffness of each member to the sums of the stiffnesses of the members at each joint, for a particular axis, is entered in Table 1 as a distribution factor. For example, the distribution factors for moment about the x-axis at joint B are $\frac{3.25}{3.25 + 1} = 0.765$ for member BA, and $\frac{1}{3.25 + 1} = 0.235$ for member BC.

As previously mentioned, a prime symbol used in connection with a carry-over factor indicates carry-over to the far end, whereas the absence of such a mark indicates carry-over to an axis at the end about which the distribution was made. That is, a moment distributed to an x-axis is multiplied by C_{xx} to obtain the moment induced about the z-axis at the same end of the member, but the distributed moment is multiplied by C'_{xx} to obtain the induced moment about the z-axis at the opposite end of the member.

To determine the fixed-end moments about the x-axis and the z-axis, at both joint A and joint B, one must first obtain the fixed-end moments about axis 1–1. These values are equal to $(\frac{1}{8})$ (20,000) 25, or 62,500 lb-ft, at each end. The fixed-end moments about the reference axes are 62,500 sin 30° = 31,250 lb-ft, about the x-axis and 62,500 cos 30° = 54,125 lb-ft, about the z-axis. These values, with appropriate signs, are entered in Table 1.

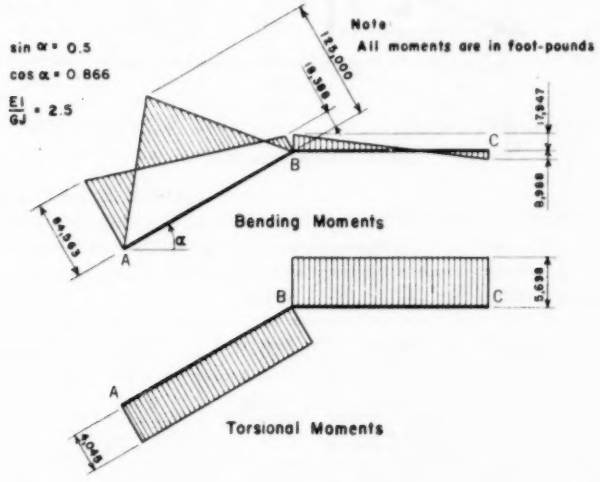
The tabulated fixed-end moments create a statical unbalance at joint B, and the unbalance about each axis is distributed to the two members at B in proportion to their relative stiffnesses about the corresponding axes. For example, the unbalanced moment of +31,250 lb-ft at joint B is distributed so that 23,906 lb-ft is resisted by member BA and 7,344 lb-ft is resisted by member BC. The distributed moments are entered in Table 1 with signs opposite to that of the unbalanced moment, and the sum of +31,250 lb-ft and −23,906 lb-ft is in balance with −7,344 lb-ft. Since joint A is fixed, there can be no unbalance at that joint; therefore, no moment is distributed to member AB at that end.

The carry-over moments (COM) are obtained by multiplying the distributed moments by the appropriate carry-over factors. Carry-over moments induced by a moment distributed to the x-axis are placed opposite the designation COM (x), and carry-over moments induced by a moment distributed to the z-axis are placed opposite the designation COM (z). To illustrate, the carry-over moments induced by the distributed moment of −23,906 lb-ft are the following: At the far end, joint A, the carry-over moments are COM(x) = − 23,906 (+0.154) = − 3,682 lb-ft about the x-axis, and COM (x) = − 23,906 (+0.799) = − 19,101 lb-ft about the z-axis; at the near end, B, COM (x) = − 23,906 (+1.199) = − 28,663 lb-ft about the z-axis. The distributed moment of −7,344 lb-ft induces a moment at the far end, C, whose value is COM (x) = − 7,344 (−1.0) = +7,344 lb-ft about the x-axis.

The carry-over moments resulting from the moments distributed to the z-axis at B are obtained in a similar manner, and are recorded opposite the designation COM (z).

At each joint, the new unbalance resulting from moments carried over is distributed in proportion to the relative stiffnesses, and the resulting values are entered in the tabulation opposite the designation "Dist." No distribution is made at the fixed ends. The procedure of successively carrying over and distributing is continued to a desired degree of convergence, and the totals are obtained. In this example the process has been terminated after a distribution. It is a matter of academic interest that the moments balance at joint B, but the moments about the x-axis at the ends of member BC are not exactly equal (5,693 lb-ft as against 5,698 lb-ft), as torsional moments should be. If the last distribution is omitted, the latter moments will balance exactly, but the moments at joint B will not.

Diagrams of bending moments and torsional moments are shown in Fig. 8(a). The bending moment diagram for AB was drawn by superimposing moments resulting from continuity on the moment diagram for beam AB considered as simply supported. Since member BC is parallel to the x-axis, the bending moments and torsional moments for that member are the moments



(a) Moment Diagrams Plotted Along the Structure

ΣF_x		ΣF_z	
+(125,000)(25)(0.5)(0.5)	= +781,250	-(125,000)(25)(0.5)(0.866)	= -1,353,125
-(84,563)(25)(0.5)(0.5)	= -528,519	+(84,563)(25)(0.5)(0.866)	= 915,394
-(18,388)(25)(0.5)(0.5)	= -114,925	+(18,388)(25)(0.5)(0.866)	= 199,050
+(4,045)(25)(2.5)(0.866)	= +218,936	+(4,045)(25)(2.5)(0.5)	= 126,406
-(5,698)(25)(2.5)	= -356,125	+(17,947)(25)(0.5)	= 224,337
	+617	-(8,988)(25)(0.5)	= -112,350
			-288

ΣM_z (about c)

-(8,988)(25)(0.5)(8.33)	= -935,875
+(17,947)(25)(0.5)(16.67)	= 3,739,706
+(18,388)(25)(0.5)(29.98)	= 6,890,903
-(125,000)(25)(0.5)(34.15)	= -53,359,375
+(84,563)(25)(0.5)(38.32)	= 40,505,677
+(4,045)(25)(2.5)(12.5)	= 3,160,156
	1,192

(b) Numerical Check

Fig. 8.—Moments and Numerical Check for the Continuous Beam of Fig. 7

about the z-axis and the x-axis, respectively, as determined in Table 1. To obtain the bending moments and torsional moments for member AB, the moments obtained in Table 1 were resolved into components about the bending axis and the axis of twist.

A solution of a statically indeterminate structure is correct providing the requirements of continuity are satisfied. To satisfy these requirements, the angle changes—as determined by means of the moments and the defined properties of the material—must "balance" around a closed circuit. By "balance" is meant that the distribution of angle changes must be such that the sum of the angle changes about any axis equals zero, and the sum of the products of angle changes times distances about any axis at a support—that is, the displacement

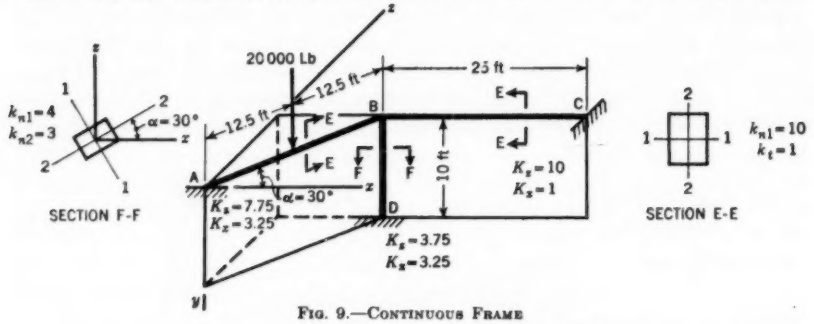


FIG. 9.—CONTINUOUS FRAME

TABLE 2.—ANALYSIS OF THE

Line No.	Description	Moments About the z-Axis, in Pound-Feet					
		Joint A	Joint B			Joint C	Joint D
		Member AB	Member BA	Member BC	Member BD	Member CB	Member DB
1	Factors: Dist.		0.433	0.134	0.433		
2	C.O.		$C'_{zz}=+0.154$ $C'_{xz}=+0.799$ $C_{zz}=+1.199$	$C'_{zz}=-1.0$	$C'_{zz}=+0.500$ $C'_{xz}=+0.067$ $C_{zz}=+0.133$		
3	F.E.M.	−31,250	+31,250				
4	Dist.		−13,531	−4,188	−13,531		
5	COM (x)	−2,084				+4,188	−6,766
6	COM (z)	−6,527	−9,801		−1,089		−549
7	Dist.		+4,715	+1,460	+4,715		
8	COM (x)	+726				−1,460	+2,358
9	COM (z)	+2,174	+3,264		+363		+183
10	Dist.		−1,570	−487	−1,570		
11	COM (x)	−241				+487	−785
12	COM (z)	−757	−1,137		−126		−64
13	Dist.		+547	+169	+547		
14	COM (x)	+84				−169	+274
15	COM (z)	+252	+379		+42		+21
16	Dist.		−182	−57	−182		
17	COM (x)	−28				+57	−91
18	COM (z)	−88	−132		−15		−7
19	Dist.		+64	+19	+64		
20	COM (x)	+10				−19	+32
21	COM (z)	+29	+44		+5		+2
22	Dist.		−21	−7	−21		
23	Totals	−37,700	+13,889	−3,091	−10,798	+3,084	−5,392

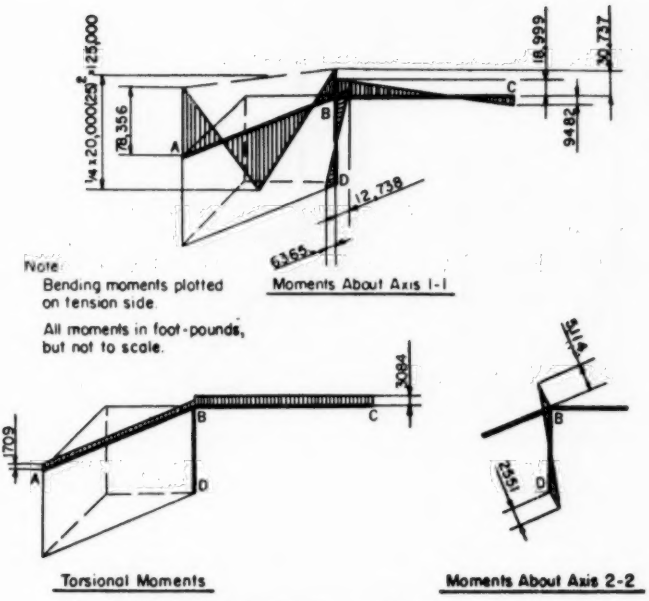
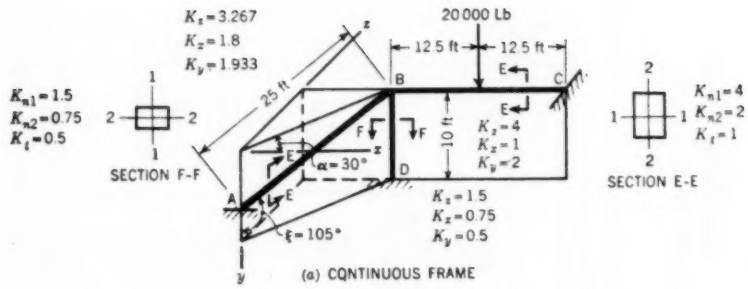


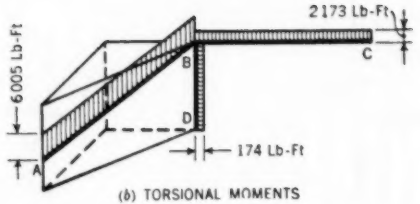
FIG. 10.—MOMENT DIAGRAMS FOR THE CONTINUOUS FRAME OF FIG. 9

CONTINUOUS FRAME OF FIG. 9

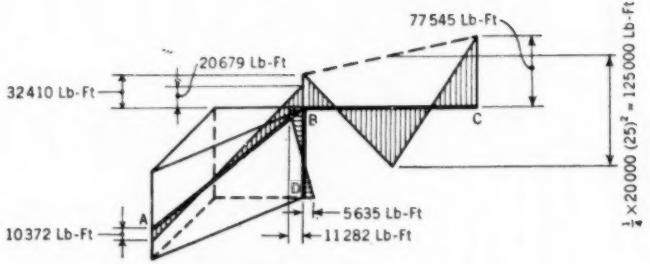
Line No.	MOMENTS ABOUT THE z-AXIS, IN POUND-FEET					
	Joint A	Joint B			Joint C	Joint D
	Member AB	Member BA	Member BC	Member BD	Member CB	Member DB
1		0.360	0.465	0.175		
2		$C'_{zz} = +0.452$ $C''_{zz} = +0.335$ $C_{zz} = +0.503$	$C'_{zz} = +0.5$	$C'_{zz} = +0.500$ $C''_{zz} = +0.058$ $C_{zz} = +0.115$		
3	−54,125	+54,125				
4		−19,485	−25,168	−9,472		
5	−10,811	−16,224		−1,800		−907
6	−8,807				−12,584	−4,736
7		+6,489	+8,381	+3,154		
8	+3,767	+5,653		+627		+316
9	+2,933				+4,190	+1,577
10		−2,261	−2,920	−1,099		
11	−1,254	−1,882		−209		−105
12	−1,022				−1,460	−550
13		+753	+972	+366		
14	+437	+656		+73		+37
15	+340				+486	+183
16		−262	−339	−128		
17	−145	−218		−24		−12
18	−118				−170	−64
19		+87	+113	+42		
20	+51	+72		+9		+4
21	+40				+56	+21
22		−29	−38	−14		
23	−68,714	+27,474	−18,999	−8,475	−9,482	−4,237



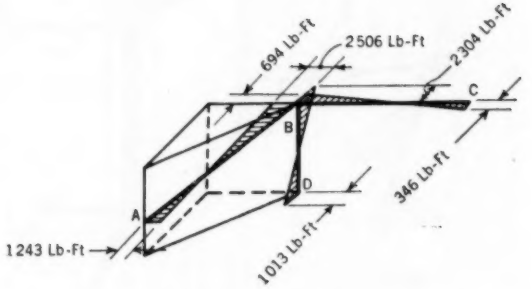
(a) CONTINUOUS FRAME



(b) TORSIONAL MOMENTS



(c) MOMENTS ABOUT AXIS 1-1



(d) MOMENTS ABOUT AXIS 2-2

FIG. 11.—CONTINUOUS FRAME AND MOMENT DIAGRAMS (BENDING MOMENTS PLOTTED ON THE TENSION SIDE)

Table 3.—Analysis of the Continuous Frame of Fig. 11

Line	Moment Joint Member	Moment About the x-Axis, in Pound-Feet						Moment About the y-Axis, in Pound-Feet						Moment About the z-Axis, in Pound-Feet						Line
		A	B			C	D	A	B			C	D	A	B			C	D	
		AB	BA	BC	BD	CB	DB	AB	BA	BC	BD	CB	DB	AB	BA	BC	BD	CB	DB	
1	Factors: Dist.		0.507	0.282	0.211				0.431	0.455	0.114				0.373	0.456	0.171			1
2	C.O.		$C'_{xx} = -0.063$ $C'_{yx} = -0.241$ $C_{xx} = +0.690$ $C_{yx} = -0.121$ $C_{zx} = +0.706$	$C'_{xx} = -1.0$	$C'_{zx} = +0.5$				$C'_{xy} = -0.225$ $C'_{yy} = +0.448$ $C'_{xy} = +0.129$ $C_{xy} = -0.112$ $C_{zy} = +0.064$	$C'_{yy} = +0.5$	$C'_{zy} = -1.0$				$C'_{xz} = +0.380$ $C'_{yz} = +0.076$ $C_{xz} = +0.394$ $C_{yz} = +0.389$ $C_{zz} = +0.038$	$C'_{xz} = +0.5$	$C'_{zz} = +0.5$			2
3	F.E.M.															−62,500		+62,500		3
4	Dist.														+23,313	+28,500	+10,688			4
5	COM (x)																			5
6	COM (y)																	+14,250	+5,344	6
7	COM (z)	+8,859	+9,069					+1,772	+886					+9,185						7
8	Dist.		−4,598	−2,557	−1,914				−382	−403	−101									8
9	COM (x)	+382				+2,557	−957	+1,108	+556					−3,173	−3,246					9
10	COM (y)	+86	+43					−171				−202	+101	−49	−24					10
11	COM (z)																			11
12	Dist.		−22	−12	−9				−240	−253	−63				+1,220	+1,491	+559			12
13	COM (x)	+18				+12	−4	+5	+3					−15	−16					13
14	COM (y)	+54	+27					−108				−126	+63	−31	−15					14
15	COM (z)	+464	+475					+93	+46					+481				+746	+274	15
16	Dist.		−255	−142	−105				−21	−22	−6				+12	+14	+5			16
17	COM (x)	+21				+142	−52	+61	+31					−176	−180					17
18	COM (y)	+5	+2					−9				−11	+6	−2	−1					18
19	COM (z)	+5	+5					+1						+5				+7	+2	19
20	Dist.		−4	−2	−1				−13	−14	−4				+68	+83	+30			20
21	COM (x)					+2		+1						−3	−3					21
22	COM (y)	+3	+1					−6				−7	+4	−2	−1					22
23	COM (z)	+26	+26					+5	+3					+27				+42	+15	23
24	Dist.		−14	−8	−5				−1	−2					+2	+2				24
25	Totals	+9,923	+4,755	−2,721	−2,034	+2,713	−1,013	+2,752	+868	−694	−174	−346	+174	+6,247	+21,129	−32,410	+11,282	+77,545	+5,635	25

at a support—equals zero. Stated algebraically,

$$\left.\begin{aligned} \Sigma d\theta_x &= 0, & \delta_x &= \Sigma z d\theta_y + \Sigma y d\theta_z = 0 \\ \Sigma d\theta_y &= 0, & \delta_y &= \Sigma x d\theta_z + \Sigma z d\theta_x = 0 \\ \Sigma d\theta_z &= 0, & \delta_z &= \Sigma x d\theta_y + \Sigma y d\theta_x = 0 \end{aligned}\right\} \ldots\ldots\ldots (7)$$

In Eqs. 7, $d\theta_x$, $d\theta_y$, and $d\theta_z$ are angle changes about the x-axis, the y-axis, and the z-axis, respectively; and δ_x, δ_y, and δ_z are displacements of one end of a circuit relative to the other end along the x-axis, the y-axis, and the z-axis, respectively.

If angle changes are treated as forces along their axes of rotation, a displacement can be computed as a moment of these "forces" about the axis of displacement. The requirements of geometry stated in Eqs. 5 can be written as analogous requirements of statics, as follows:

For rotations:	For displacements:
$\Sigma F_x = 0,$	$\Sigma M_x = 0$
$\Sigma F_y = 0,$	$\Sigma M_y = 0$
$\Sigma F_z = 0,$	$\Sigma M_z = 0$

$\ldots\ldots\ldots\ldots\ldots\ldots (8)$

In Fig. 8(b) a check is made of the solution through application of the appropriate relationships from Eqs. 8. Angle changes were first obtained about the principal axis of bending and the axis of twist using the moments in Fig. 8(a) and the appropriate values of bending and torsional stiffness per unit length of a member. Such angle changes are obtained by dividing bending moments by $E\,I$ and torsional moments by $G\,J$. It is only necessary to use relative values of $E\,I$ and $G\,J$. The angle changes about the x-axis and the z-axis ("forces" in the x and z directions, respectively) were obtained by use of the appropriate functions of angle α. In Fig. 8(b), all values of angles are based on relative values of $E\,I$ and $G\,J$ equal to 2.5 lb-sq ft and 1.0 lb-sq ft, respectively. The results of the check in 8(b) show that the requirements of geometry are satisfied.

Except for the addition of column BD, the structure shown in Fig. 9 and analyzed in Table 2 is identical to that in Fig. 7. The column is so oriented that the principal axis 2–2 of the section is parallel with the longitudinal axis of member AB. From the expressions in Fig. 6 ($\zeta = 0$), the stiffnesses and carry-over factors for BD were determined. The numerical procedure follows in a manner similar to that in Table 1. Moment diagrams are shown in Fig. 10. A check of the angle changes along the circuit ABC and along the circuit DBC showed an almost perfect balance.

In the frame in Fig. 11 (a), member AB does not lie in any of the three orthogonal planes. The projected angle α of this member is equal to 30°, the angle ζ is equal to 105°, and the angle ψ is equal to zero. Column BD is so oriented that the principal axes of its cross section are parallel to the chosen reference axes, thus making angle α for this member equal to zero. Girders AB and BC are identical in cross section. Values of stiffnesses and carry-over factors for member AB were determined through use of the general expressions in Fig. 4. Because the principal axes of the cross sections of members BC and BD are parallel to the reference axes, their stiffnesses about the latter axes are identical to those about the corresponding principal axes. The structure is analyzed in Table 3.

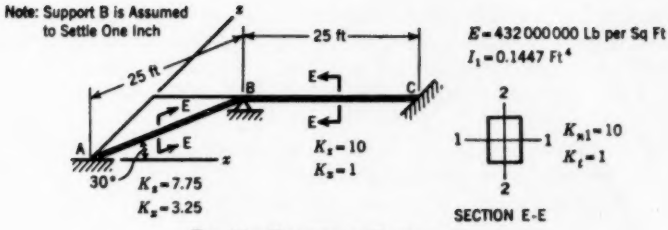


FIG. 12.—EFFECTS OF TRANSLATION

TABLE 4.—ANALYSIS OF THE EFFECTS OF TRANSLATION OF THE CONTINUOUS BEAM OF FIG. 12

DESCRIPTION	MOMENTS ABOUT THE x-AXIS, IN POUND-FEET				MOMENTS ABOUT THE z-AXIS, IN POUND-FEET			
	Joint A	Joint B		Joint C	Joint A	Joint B		Joint C
	Member AB	Member BA	Member BC	Member CB	Member AB	Member BA	Member BC	Member CB
Factor:		0.765	0.235			0.437	0.563	
Dist.		$C'_{xx} = +0.154$	$C'_{xz} = -1.0$			$C'_{zz} = +0.452$	$C'_{zz} = +0.5$	
C.O.		$C'_{zx} = +0.799$				$C'_{xz} = +0.335$		
		$C_{xx} = +1.199$				$C_{zz} = +0.503$		
F.E.M.	−25,000	−25,000			−43,300	−43,300	+50,000	+50,000
Dist.		+19,125	+5,875			−2,928	−3,772	
COM (x)	+2,945			−5,875	+15,281	+22,931		
COM (z)	−981	−1,473			−1,323			−1,886
Dist.		+1,127	+346			−10,021	−12,910	
COM (x)	+174			−346	+900	+1,351		
COM (z)	−3,357	−5,041			−4,529			−6,455
Dist.		+3,856	+1,185			−590	−761	
COM (x)	+594			−1,185	+3,081	+4,693		
COM (z)	−198	−297			−267			−380
Dist.		+227	+70			−2,051	−2,642	
COM (x)	+35			−70	+181	+272		
COM (z)	−687	−1,032			−927			−1,321
Dist.		+789	+243			−119	−153	
COM (x)	+122			−243	+630	+946		
COM (z)	−40	−60			−54			−76
Dist.		+46	+14			−413	−533	
COM (x)	+7			−14	+37	+55		
COM (z)	−138	−208			−187			−266
Dist.		+159	+49			−24	−31	
COM (x)	+24			−49	+127	+191		
COM (z)	−8	−12			−11			−16
Dist.		+9	+3			−86	−108	
Totals	−26,508	−7,785	+7,785	−7,782	−30,361	−29,093	+29,090	+39,600

Diagrams of bending moments and torsional moments for the structure are shown in Fig. 11. Again a check of the angle changes along the circuits ABC and DBC showed a very good balance. Of course, in this case it was necessary to check the balance for angle changes ("forces") associated with the x, z, and y directions.

The analysis of a structure in which joint translation occurs is illustrated in Fig. 12. The structure considered is that of Fig. 7, and it is assumed that the support B settles 1 in. Fixed-end moments, about axis 1–1, are equal to $6\,E\,I\,\frac{\Delta}{L^2} = 6 \times 432{,}000{,}000 \times 0.1447 \times \frac{1/12}{-(25)^2} = 50{,}000$ lb-ft. Fixed-end moments about the reference axes were next obtained and entered in Table 4, and the procedure was applied as before.

Summary

A procedure has been presented for the structural analysis of girders and frames that are continuous in space. As herein presented, it is applicable to structures composed of straight members; the members may be of either constant or variable cross section. The method involves successive corrections of an assumed solution, and consequently lends itself to approximate or precise analyses.

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